Automatic Control Engineering

Chapter Four

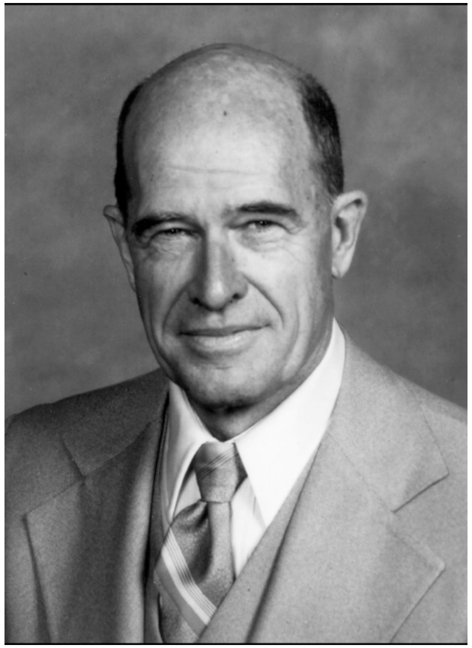
Root Locus

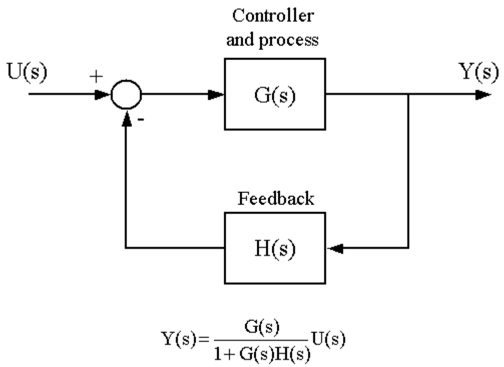
The effect of parameter variation
in the behavior of systems

1. Introduction

In previous chapters, systems have been analyzed assuming constant system parameters. But this is not always the case of dynamic systems. Parameters are subjected to variations (even small or large) because of: variety of settings available and external effects on parameters (temperature, moisture, magnetic fields, age, dust, …etc). These variations and setting changes reflects on the dynamic behavior through the locations of roots of the characteristic equation on the complex plane, hence stability and its modes. A path that the roots are moving through is called root locus (plural is loci).

The root locus concept, also known as “Evans’ rules” in honor of W. R. Evans, is a technique for determining how the poles of a feedback control system move in the complex plane as a parameter is varied. Typically, the parameter is a control gain (K), although any parameter of interest can be used. (For this reason, the root locus method is useful in dynamical system theory, where one is often interested in sudden changes in a system’s qualitative behavior, called “bifurcations,” as a parameter varies.)





Consider a simple feedback control system
its closed-loop transfer function is

C(s)/R(s)=G(s)/(1+G(s)H(s))

Closed-loop poles are values of s,
(or roots of the characteristic equation), for which

$1 + G(s)H(s) = 0.$

Since G(s)H(s) is a function of a complex variable, the equation

$G(s)H(s) = -1$

can be expressed in terms of the magnitude and phase of G(s)H(s):

$|G(s)H(s)| = 1$ and $\angle G(s)H(s) = (2k + 1)\pi$, $k = 0,\pm1,\pm2, \ldots$

In words, the magnitude of the “open loop gain” is always one and the phase is an odd power of π.

Suppose that the transfer function (TF) can be written in the form

$TF(s)=K\, G_1(s)/(1+G(s)\, H_1(s))$

This would be the case, for example, if $TF(s) = Pb(s)/a(s)$ and $P = K$, as for a simple proportional controller. The control structure might be more complicated than this, however we assume that a the multiplicative factor K appears and that this parameter may vary.

The “root locus” is the “locus” of possible roots of the closed-loop transfer function as the multiplicative parameter K is varied. In fact, the entire root locus can be determined from the angle condition alone. The magnitude condition is then used to determine which value of K corresponds to which set of closed-loop poles along the locus of all possible closed-loop poles.

Example 1. To begin, we consider the very simple example

$G(s)H(s) = K/(s(s+2))$

.

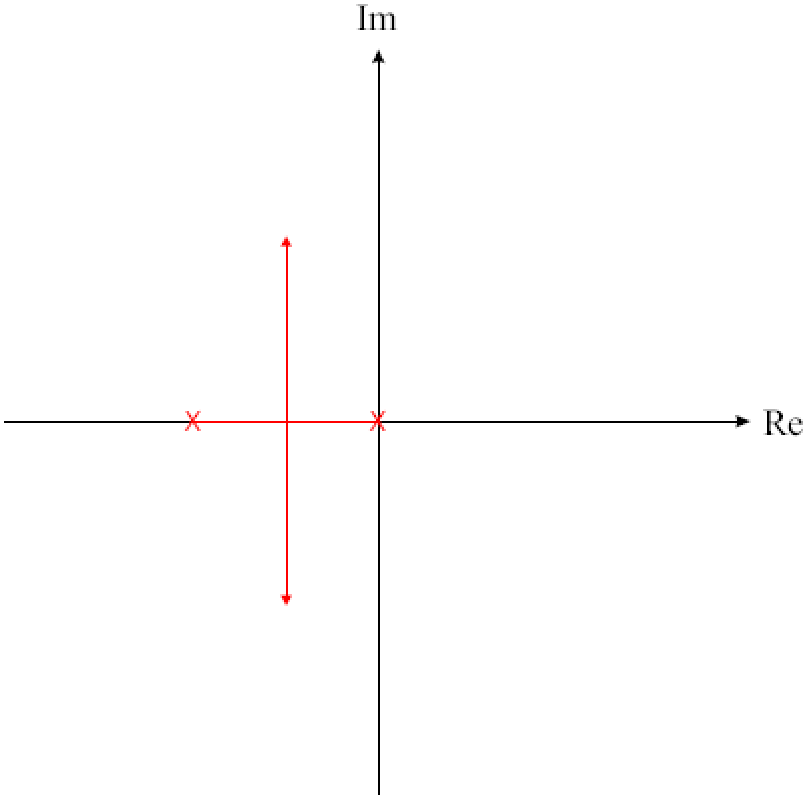
We will compute the closed-loop poles as explicit functions of K. In general, this is a tedious, and uninformative exercise, but for this simple system it serves to illustrate how closed-loop poles vary as the gain K is varied. The characteristic equation is

$1+G(s)H(s)=0$
$s^2 + 2s + k = 0$

The closed-loop poles (roots of characteristic equation) are
$S_{1,2} = -1 \pm \mathrm{sqrt}(1 - K)$

When $0 < K < 1$, there are two distinct poles which are located on the real axis between 0 and −2. When $K = 1$, the poles coalesce at $s = -1$.

As K continues to increase, the poles split apart and move in opposite directions parallel to the imaginary axis. The locus of the poles as K goes from 0 to ∞ is shown below:



Evan's rules

Following is the general procedure for constructing a root locus plot using Evan's rules, as adapted from Ogata's book[ISBN81-7808-579-8].

Rule 1. Locate the poles and zeros of the loop gain G(s)H(s). First, compute the zeros of the loop gain (the roots of numerator)) and place an 'o' at their location in the complex plane. Next, compute the poles of the loop gain (the roots of denominator) and place an 'x' at their location in the complex plane. Each portion of the root locus starts from a pole and ends to a zero. So, at poles, $K=0$ and at zeros, $K=\infty$.

Rule 2. Determine what, if any, portion of the real axis is part of the root locus. The angle condition requires that the real axis portion of the root locus lies to the left of an odd number of poles and zeros. Equivalently, since complex poles and zeros must occur in conjugate pairs, the real axis portion of the root locus lies to the left of an odd number of real poles and zeros.

Rule 3. Determine the asymptotes of the root locus. Given that there are m zeros and n poles, m of the closed-loop poles will approach the loop gain zeros as K →∞ and the remaining n – m will converge to asymptotes which extend radially to infinity from some starting point on the real axis.

The asymptote angles are

$$\theta_k = \frac{(2k+1)\pi}{n-m} \qquad k = 0, \pm 1, \pm 2, \ldots,$$

The center of the asymptotes (centroid) can be computed from:

$$\sigma = \frac{(p_1 + \cdots + p_n) - (z_1 + \cdots + z_m)}{n-m}.$$

Rule 4. Determine the angles of departure from the loop gain poles and the angles of arrival at the loop gain zeros. Recall that as K → 0, the root locus approaches the poles of the loop gain and as K →∞, m branches of the root locus approach the zeros of the loop gain.

The angle of departure from the kth loop gain pole p_k can be obtained from the angle condition as:

$$\theta_d = \pi + \sum_i \angle(p_k - z_i) - \sum_{j \neq k} \angle(p_k - p_j).$$

That is, the departure angle is π plus the sum of all the angles of vectors pointing from the loop gain zeros to p_k minus the sum of all the angles of vectors pointing from the remaining loop gain poles to p_k.

of :Similarly, one can use the angle condition to show that the angle arrival at the kth loop gain zero z_k is

$$\theta_a = \pi - \sum_{i \neq k} \angle(z_k - z_i) + \sum_j \angle(z_k - p_j).$$

Rule 5. Intersection with Imaginary axis: in general, it is a good idea to also compute the value of K at which the root locus crosses into the right half of the complex plane for the first time. This can be done using the Routh-Hurwitz procedure. The gain value at which the root locus first crosses into the right half plane generally serves as an upper limit on acceptable choices of the parameter K

Rule 6. Find the breakaway and breakin points. Recall that these points correspond to values of the gain K for which the closed-loop system has multiple closed-loop poles at a particular point. For a double-pole, the condition

$$\frac{\mathrm{d}}{\mathrm{d}s}[1 + G_0(s)] = \frac{\mathrm{d}}{\mathrm{d}s}G_0(s) = 0 .$$

or,

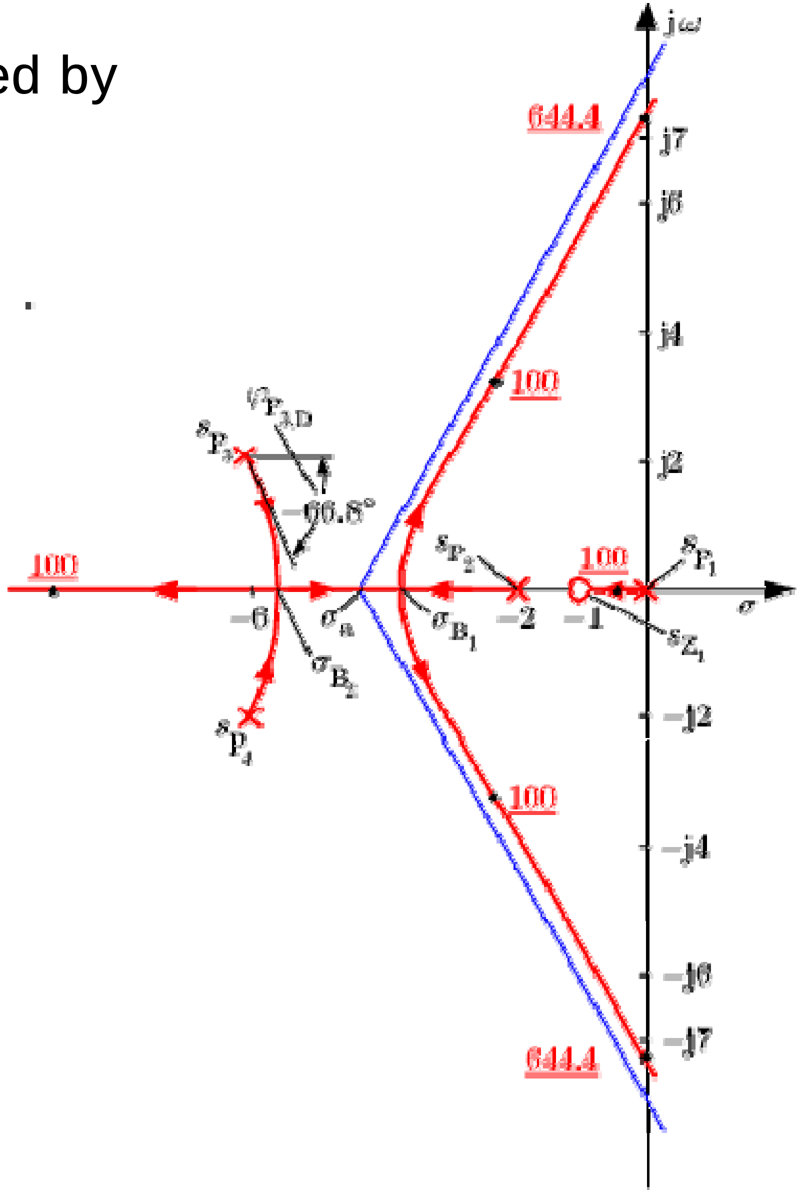
$$\sum_{\nu=1}^{n} \frac{1}{s - sp_{\nu}} - \sum_{\mu=1}^{m} \frac{1}{s - sz_{\mu}}$$

must be satisfied. The roots of this algebraic equation give possible breakaway or breakin points. To determine whether these are, in fact, breakaway or break in points, one must check whether these points are actually on the root locus.

Example: A system is represented by an open loop TF as,

$$G_0(s) = \frac{k_0(s+1)}{s(s+2)(s^2+12s+40)}\ .$$

Using Evan’s rules, its root locus is constructed as shown in the plot.



n

Rule 1. Symmetry :As all roots are either real or complex conjugate pairs so that the root locus is symmetrical to the real axis.

Rule 2. Number of branches :The number of branches of the root locus is equal to the number of poles of the open-loop transfer function.

Rule 3. Locus start and end points :The locus starting points () are at the open-loop poles and the locus ending points ( ) are at the open-loop zeros. branches end at infinity. The number of starting branches from a pole and ending branches at a zero is equal to the multiplicity of the poles and zeros, respectively. A point at infinity is considered as an equivalent zero of multiplicity equal to .