Control Engineering
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sheet # 4
State Space Analysis

Question. Consider a control system with state model,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$x_1(0) = 0,\ x_2(0) = 1,\ u = \text{unit step}$

Compute the state transition matrix and find the time response.

Solution. We have,

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\ X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[SI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\text{Adj } A = \begin{pmatrix} s+3 & -2 \\ 1 & s \end{pmatrix}^T = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$[SI - A]^{-1} = \frac{\text{Adj } A}{|SI - A|}$$

$$= \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix}} = \frac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{s+1} - \frac{1}{s+2}\right) & \left(\frac{1}{s+1} - \frac{1}{s+2}\right) \\ \left(\frac{-2}{s+1} + \frac{2}{s+2}\right) & \left(\frac{-1}{s+1} + \frac{2}{s+2}\right) \end{bmatrix}$$

$$e^{At} = \Phi(t) = \mathcal{L}^{-1}[SI - A]^{-1}$$

$$= \begin{bmatrix} (2e^{-t} - e^{-2t}) & (e^{-t} - e^{-2t}) \\ (-2e^{-t} + 2e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}$$

The time response of the linear time-invariant, non-homogeneous system is given by,

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

for unit step input, $u(t) = 1$ for $t > 0$,

$$x(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (e^{-t} - e^{-2t}) \\ (-2e^{-t} + 2e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} +$$

$$\int \begin{bmatrix} 2e^{(t-\tau)} - e^{2(t-\tau)} & e^{(t-\tau)} - e^{2(t-\tau)} \\ -2e^{(t-\tau)} + 2e^{2(t-\tau)} & (-e^{(t-\tau)} + 2e^{(t-\tau)}) \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot d\tau$$

$$= \begin{bmatrix} (e^{-t} - e^{-2t}) \\ (-e^{-t} + 2e^{-2t}) \end{bmatrix} + \int \begin{bmatrix} 2\left(e^{(t-\tau)} - e^{2(t-\tau)}\right) \\ 2\left(-e^{(t-\tau)} - e^{2(t-\tau)}\right) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} - e^{-2t} + \int 2e^{(t-\tau)} \cdot d\tau - \int 2e^{2(t-\tau)} \cdot d\tau \\ -e^{-t} + 2e^{-2t} - \int_0^t 2e^{(t-\tau)} \cdot d\tau - \int_0^t 2e^{2(t-\tau)} \cdot d\tau \end{bmatrix}$$

To solve the integral part, we can write

$$\int_0^t e^{A(t-\tau)} B\, u(\tau) \cdot d\tau$$

$$= e^{At} \int_0^t e^{A(-\tau)} B\, u(\tau) \cdot d\tau, \text{ for } u(t) = 1;\ t > 0,$$

$$e^{A(-\tau)} B\, u(\tau) = \begin{bmatrix} (2e^{\tau} - e^{2\tau}) & (e^{\tau} - e^{2\tau}) \\ (-2e^{\tau} + 2e^{2\tau}) & (-e^{\tau} + 2e^{\tau}) \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} (2e^{\tau} - 2e^{2\tau}) \\ (-2e^{\tau} + 4e^{2\tau}) \end{bmatrix}$$

$$\therefore \int_0^t e^{A(\tau)} B\, u(\tau) \cdot d\tau = \begin{bmatrix} (2e^{\tau} - e^{2\tau}) \\ (-2e^{\tau} + 2e^{2\tau}) \end{bmatrix}_0^t = \begin{bmatrix} 2e^{t} - e^{2t} \\ -2e^{t} - 2e^{2t} \end{bmatrix} - \begin{bmatrix} 2 - 1 \\ -2 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{t} - e^{2t} - 1 \\ -2e^{t} + 2e^{2t} \end{bmatrix}$$